QP CODE: 21000687

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Reg No : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, JULY 2021

Fourth Semester

Faculty of Science

CORE - ME010402 - ANALYTIC NUMBER THEORY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

B052CF42

Time: 3 Hours | Weightage: 30

Part A (Short Answer Questions)

Answer any **eight** questions.

Weight **1** each.

1. Define Euler Totient function $\phi(n)$. Also prove that $\phi(n)$ is even for $n \geq 3$.
2. State Euler's summation formula and define Riemann zeta function.
3. Explain the mutual visible lattice points. State a necessary and sufficient condition for two lattice points (a,b) and (m,n) to be mutually visible.
4. Derive Euler's summation formula from Abel's identity.
5. Write any four equivalent forms of the prime number theorem.
6. (a) If $ac \equiv bc (mod\, m)$ and if $d = (m,c)$, then prove that $a \equiv b(mod \frac{m}{d})$.
   (b) If $c > 0$ then prove that $a \equiv b(mod\, m)$ if and only if $ac \equiv bc(mod\, mc)$.
7. Define residue class a *modulo* m and prove that for a given modulus m the m residue classes $\hat{1}, \hat{2}, \ldots, \hat{m}$ are disjoint and their union is the set of all integers.
8. If $\{a1, a2, \ldots a_{\phi(m)}\}$ is a reduced residue system modulo m and if $(k,m) = 1$ then prove that $\{ka1, ka2, \ldots ka_{\phi(m)}\}$ is also a reduced residue system modulo m.
9. Define quadratic residues. Find the quadratic nonresidues for p = 13.
10. (a) Define $exp_m(a)$.
    (b) Let $m \geq 1$ and $(a,m) = 1$. Then prove that $a^k \equiv a^h(mod\, m)$ if and only if $k \equiv h(mod\, m)$,where$f = exp_m(a)$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

Answer any **six** questions.

Weight **2** each.

11. Prove that if both g and $f * g$ are multiplicative then f is multiplicative.

12. (a) Prove that if f is multiplicative then $\sum_{d|n}\mu(d)f(d) = \Pi_{p|n}(1 - f(p))$.

(b)State and prove the associative property relating $\circ$ and $*$.

13. Show that the n^{th}prime P_n satisfies the inequality $\frac{1}{6}n \log n < P_n < 12(n \log n + n \log\frac{12}{e}), \forall n \geq 1$.

14. Show that $(i) \sum_{n\leq x}\psi(\frac{x}{n}) = x \log x - x + O(\log x)$ and $(ii) \sum_{n\leq x}\vartheta(\frac{x}{n}) = x \log x - x + O(x)$.

15. Given a prime p,let $f(x) = c_0 + c_1x + \ldots c_nx^n$ be a polynomial of degree n with integer coefficents such that $c_n \not\equiv 0(mod\, p)$. Then prove that polynomial congruence $f(x) \equiv o(mod\, p)$has at most n solutions.

16. Find all x which simultaneously satisfy the system of congruences $x \equiv 1(mod\, 3), x \equiv 2(mod\, 4), x \equiv 3(mod\, 5)$.

17. Prove that $(-1|p) = -1$ if $p = 4m + 3$for some integer m. Also write a formula for $(2|p)$when p is an odd prime.

18. Let g be a primitive root mod p, where p is an odd prime. Then prove that the even powers $g^2, g^4, \ldots, g^{p-1}$are the quadratic residues and the odd powers $g, g^3, \ldots, g^{p-2}$are the quadratic nonresidues mod p.

(6×2=12 weightage)

Part C (Essay Type Questions)

Answer any **two** questions.

Weight **5** each.

19. (a)For $x \geq 1$ prove that $\left|\sum_{n\leq x}\frac{\mu(n)}{n}\right| \leq 1$ with equality holding only if $x < 2$.

(b)Prove that for every $x \geq 1$, $[x]! = \Pi_{p\leq x}p^{\alpha(p)}$where the product is extended over all primes $\leq x$, and $\alpha(p) = \sum_{m=1}^{\infty}[\frac{x}{p^m}]$.

(c) If $x \geq 2$,prove that $log[x]! = x \log x - x + O(\log x)$.

20. Let $\{a(n)\}$ be a nonnegative sequence such that $\sum_{n\leq x}a(n)[\frac{x}{n}] = x \log x + O(x) for\ all\ x \geq 1$. Then prove the following

(a) $\forall x \geq 1, we\ have \sum_{n\leq x}\frac{a(n)}{n} = \log x + O(1)$.

(b) There is a constant B such that $\sum_{n\leq x}a(n) \leq Bx, \forall x \geq 1$.

(c) There is a constant $A > 0$and an $x_0 > 0$ such that $\sum_{n\leq x}a(n) \geq Ax, \forall x \geq x_0$.

21. (a) Prove that for a given integer $k > 0$ there exist a lattice point (a, b) such that none of the lattice points $(a + r, b + s)$,$0 < r \leq k$,$o < s \leq k$, isvisible from the origin.

(b) Let f be a polynomial with integer coefficients, let $m_1, \ldots, m_r$ be positive integers relatively prime in pairs, and let $m = m_1m_2\ldots m_r$.Prove that the congruence $f(x) \equiv 0(mod\, m)$ has a solution if and only if each of the congruences $f(x) \equiv o(mod\, m_i)$ $(i = 1, \ldots, r)$ has a solution. Also show that if $v(m)$ and $v(m_i)$ denote the number of solutions of $f(x) \equiv 0(mod\, m)$and $f(x) \equiv o(mod\, m_i)$for $i = 1, \ldots, r$, respectively, then $v(m) = v(m_1)v(m_2)\ldots v(m_r)$.

22. Assume n is not congruent to $0(modp)$ and consider the least positive residues mod p of the following $\frac{p-1}{2}$multiples of $n : n, 2n, 3n, \ldots, \frac{p-1}{2}n$. Then if m denotes the number of these residues which exceed $\frac{p}{2}$, prove that $(n|p) = (-1)^m$.

(2×5=10 weightage)