QP CODE: 22000350

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Reg No : ....

Name : ....

MSc DEGREE (CSS) EXAMINATION , JANUARY 2022

Second Semester

CORE - ME010201 - ADVANCED ABSTRACT ALGEBRA

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

44499036

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. *Prove that the set of all algebraic numbers form a field.*
2. *Prove that a finite extension E of a finite field F is a simple extension of F.*
3. Express $18x^2 - 12x + 48$ as a product of its content with a primitive polynomial in $\mathbb{Z}[x]$
4. Check whether the function ν for the integral domain $\mathbb{Z}$ given by $\nu(n) = n^2$ for nonzero $n \in \mathbb{Z}$ is a Euclidean norm.
5. Define Gaussian integers and a norm for it.
6. *Prove that for* $a, b \in \mathbb{R}$ *with* $b \neq 0$, *the conjugate complex numbers* $a + bi$ *and* $a - bi$ *are conjugate over* $\mathbb{R}$.
7. *What is the order of* $G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$?
8. *Prove that the splitting field over* $\mathbb{Q}$ *of* $x^3 - 2$ *is of degree* 6 *over* $\mathbb{Q}$.
9. *Let* $f(x)$ *be a polynomial in* $F[x]$ *where* F *is a field. Define the group of* $f(x)$ *over* F.
10. *Show that* $x^4 + 1$ *is irreducible in* $\mathbb{Q}[x]$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Find the degree and a basis for $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{18})$ over $\mathbb{Q}$
12. *If* α *and* β *are constructible real numbers, then prove that* $\alpha + \beta, \alpha - \beta, \alpha\beta, \alpha/\beta$ *when* $\beta \neq 0$ *are constructible.*
13. Define an irreducible element in a PID. Prove that an ideal $\langle p \rangle$ in a PID is maximal if and only if p is an irreducible.

14. Define(i) UFD, (ii) PID, (iii) Euclidean domain

15. *Let* $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ *and* $F = \mathbb{Q}$. *Let* $\sigma_1 = \psi_{\sqrt{2},-\sqrt{2}}$, $\sigma_2 = \psi_{\sqrt{3},-\sqrt{3}}$ *and* $\sigma_3 = \sigma_1\sigma_2$. *Find the fixed fields* $E_{\{\sigma_1,\sigma_3\}}$, $E_{\{\sigma_3\}}$ *and* $E_{\{\sigma_2,\sigma_3\}}$.

16. *Let* E *be a finite extension of a field* F. *Let* σ *be an isomorphism of* F *onto a field* F' *and let* $\overline{F'}$ *be an algebraic closure of* F'. *Prove that the number of extensions of* σ *to an isomorphism* τ *of* E *onto a subfield of* $\overline{F'}$ *is finite and independent of* F', $\overline{F'}$ *and* σ.

17. Let $\overline{F}$ be an algebraic closure of a field F and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a monic polynomial in $\overline{F}[x]$. *If* $(f(x))^m \in F[x]$ *and* $m{\cdot}1 \neq 0$ *in* F, *prove that* $f(x) \in F[x]$, *that is, all* $a_i \in F$.

18. State and prove Primitive element theorem.

(6×2=12 weightage)

Part C (Essay Type Questions)

Answer any ***two*** *questions.*

Weight ***5*** *each.*

19. a) Let F be a field and let f(x) be a nonconstant polynomial in F[x]. Then prove that there exists an extension field E of F and an $\alpha \in E$ such that $f(\alpha)=0$

    b) Construct a finite field of 4 elements

20. a) If D is a UFD, then prove that a product of two primitive polynomials in D[x] is again primitive.

    b) Let D be a UFD and let F be a field of quotients of D. Let f(x) in D[x] has degree greater than 0. If f(x) is irreducible in D[x], then prove that f(x) is also irreducible in F[x]. Also if f(x) is primitive in D[x] and irreducible in F[x], then prove that f(x) is irreducible in D[x].

21. a) State and prove the isomorphism extension theorem.

    b) Prove that any two algebraic closures of a field F *are isomorphic under an isomorphism leaving each element of* F *fixed.*

22. a) Let F be a field and $f(x)$ be an irreducible polynomial in $F[x]$. Prove that all zeros of $f(x)$ in $\overline{F}$ have the same multiplicity.

    b) Let F be a field and $f(x)$ be an irreducible polynomial in $F[x]$. Prove that $f(x)$ has a factorization in $\overline{F}[x]$ of the form $a\prod_i(x-\alpha_i)^\nu$ where α_i are the distinct zeros of $f(x)$ in $\overline{F}$ and $a \in F$.

    c) If E is a finite extension of a field F, then prove that $\{E : F\}$ divides $[E : F]$.

(2×5=10 weightage)