QP CODE: 22001756

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Reg No : ....

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M Sc DEGREE (CSS) EXAMINATION, AUGUST 2022

Fourth Semester

Core - ME010401 - SPECTRAL THEORY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

DC24216F

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define strong and weak convergence of a sequence in a normed space. Prove that strong convergence implies weak convergence with the same limit.

2. If $S_n, T_n \in B(X,Y)$, and (S_n) and T_n are strongly operator convergent with limits S and T, Show that $(S_n + T_n)$ is strongly operator convergent with the limit $S+T$.

3. Define the spectral radius $r_\sigma(T)$ of an operator $T \in B(X,X)$, where X is a Banach space. Also write down the expression for finding $r_\sigma(T)$.

4. Let $T \in B(X,X)$, where X complex Banach space and $\mu, \lambda \in \rho(T)$. Then prove that $R_\lambda R_\mu = R_\mu R_\lambda$.

5. Let matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where a, b, c, d are real numbers and $ad - bc \neq 0$. If $\{2, 3\}$ is the spectrum of A, find the spectrum of B.

6. Define Banach algebra with example.

7. Show that the resolvent $\rho(x)$ is open, where $x \in A$ and A is a complex Banach algebra with identity.

8. If T is a compact linear operator on a normed space X prove that the range of $T_\lambda{}^r$ is closed for every $\lambda \neq 0$.

9. *Define self-adjoint linear operator on a Hilbert space. Prove that the eigen vectors corresponding to distinct eigen values of a bounded self-adjoint linear operator on a complex Hilbert space are orthogonal.*

10. Let T be a bounded self-adjoint linear operator on a Hilbert space H. Show that if $T \geq 0$, then $(I+T)^{-1}$ exists.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let $T_n : l^2 \to l^2$ be a sequence of operators defined as

$$T_n(x) = (\underbrace{0, 0, \cdots, 0}_{\text{n zeros}}, \xi_1, \xi_2, \xi_3, \cdots)$$

where $x = (\xi_1, \xi_2, \cdots) \in l^2$. Show that

(a) T_n is linear and bounded.

(b) T_n is weakly operator convergent to 0 but not strongly.

12. Let X and Y be Banach spaces and $T : \mathcal{D}(T) \to Y$ a closed linear operator, where $\mathcal{D}(T) \subset X$. Prove that if $\mathcal{D}(T)$ is closed in X, then the operator T is bounded.

13. Define eigenvalues of a linear operator $T : D(T) \to X$, where $X \neq \{0\}$ is a complex normed space and $D(T) \subset X$. Also, give an example for a linear operator having spectral values which are not eigenvalues. Justify your answer.

14. Let $T : X \to X$ be a bounded linear operator on a complex Banach space X. Prove that the resolvent operator $R_\lambda(T)$ is holomorphic at every point $\lambda_0 \in \rho(T)$.

15. Show that $T : l^2 \to l^2$ defined by $Tx = y = (\eta_j), \eta_j = \frac{\xi_j}{2}$ is compact, where $x = (\xi_j), j = 1, 2, 3, \cdots$.

16. If B is a totally bounded subset of a metric space X, prove that B contains a finite ϵ - net for every $\epsilon > 0$.

17. Prove that the spectrum of a bounded self-adjoint linear operator on a complex Hilbert space lies in a closed interval on the real axis.

18. Show that the difference $P = P_2 - P_1$ of two projections on a Hilbert space H is a projection on H if and only if $P_1 \leq P_2$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. (a) Prove that a bounded linear operator T from a Banach space X onto a Banach space Y has the property that the image $T(B_0)$ of the open unit ball $B_0 = B(0; 1) \subset X$ contains an open ball about $0 \in Y$.

(b) State and prove Open Mapping Theorem.

20. Let $T : X \to X$ be a contraction on a complete metric space $(X, d); X \neq \phi$. Prove that T has precisely one fixed point.

21. Let $T : X \to X$ be a compact linear operator on a Banach space X, and $\lambda \neq 0$. Prove that there exists a smallest integer r such that from $n = r$ onwards the null spaces $\mathcal{N}(T_\lambda^n)$ are all equal and if $r > 0$, the inclusions $\mathcal{N}(T_\lambda^0) \subset \mathcal{N}(T_\lambda) \subset \mathcal{N}(T_\lambda^2) \subset \cdots \subset \mathcal{N}(T_\lambda^r)$ are all proper.

22. State and prove a necessary and sufficient ondition for a projection on a Hilbert space H.

(2×5=10 weightage)