QP CODE: 23003110

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Reg No : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, APRIL 2023

First Semester

CORE - ME010102 - LINEAR ALGEBRA

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

798F2CA1

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define vector space. Is $\mathbb{C}$ a vector space over $\mathbb{R}$?
2. Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, and γ are linearly independent vectors in V. Prove that *(α + β), (β + γ), and (γ + α) are linearly independent.*
3. Find T^2 where $T: R^2 \to R^2$ is a linear operator defined as $T(x_1, x_2) = (x_2, x_1)$.
4. Show that the space C of complex numbers and R^2 are isomorphic, considering as vector spaces over R.
5. If $T: C^2 \to C^2$ is a linear operator defined as $T(x_1, x_2) = (x_1, 0)$,find the matrix of T in the standard ordered basis for C^2.
6. *Define commutative ring. Give examples for commutative and non-commutatve rings.*
7. *If K is a commutative ring with identity and A is the matrix over K given by A =* $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ *then find det A.*
8. Show that if two matrices are similar, then their determinants are the same.
9. *Prove that similar matrices have the same characteristic values*
10. *Find a* 3×3 *matrix for which the minimal polynomial is* x^2.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let V be the vector space of all 2 × 2 matrices over the field $\mathbb{C}$ of complex numbers. Let W_1 be the subset of V consisting of all matrices of the form $\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$ and let W_2 be the subset of V consisting of all matrices of the form $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$.

    (i) Prove that W_1 and W_2 are subspaces of V.

    (ii) Show that $W_1 + W_2 = V$.

    (iii) Find $W_1 \cap W_2$. Is it a subspace? If so, find its dimension.

12. *Show that the vectors* $\alpha_1 = (\cos\theta, \sin\theta), \alpha_2 = (-\sin\theta, \cos\theta)$ *form a basis for* $\mathbf{R}^2$. *Find the coordinates of each of the standard basis vectors in the ordered basis* $\{\alpha_1, \alpha_2\}$.

13. If W_1 and W_2 are sub spaces of a finite dimensional vector space V then prove that

    (i) $W_1 \subset W_2 \Rightarrow W_2^0 \subset W_1^0$

    (ii) $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$

14. Let V be a vector space of any dimension over the field F. Prove that every hyper space in V is the null space of a non-zero linear functional on V.

15. Let V and W be vector spaces over the field F and $T : V \to W$ is a linear transformation. Prove that null space of T^t is the annihilator of the range of T.

16. *Let n > 1 and let D be an alternating (n-1)-linear function on (n -1) x (n -1) matrices over K. For each j, 1 ≤ j≤ n, show that the function E_j defined by* $$\text{Ej}(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)$$ *is an alternating n-linear function on n x n matrices A.*

17. *Let* T *be a linear operator on* $\mathbb{R}^4$ *which is represented in the standard ordered basis by the matrix*
$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$
. Under what conditions on a, b and c, T *is diagonalizable.*

18. *Let* V *be a finite dimensional vector space and let* $W_1, \cdots, W_k$ *be subspaces of V such that* $V = W_1 + \cdots + W_k$ *and* $\dim W_1 + \dim W_2 + \cdots + \dim W_k = \dim V$ *Prove that* $V = W_1 \oplus \cdots \oplus W_k$

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. (a) Show that row-equivalent matrices have the same row space.

    (b) Let R be a non-zero row-reduced echelon matrix. Then prove that the non-zero row vectors of R form a basis for the row space of R.

20. a) If A is an $n \times n$ matrix with entries in the field F. Then prove that the row rank and column rank of A are equal.

    b) Determine the row rank of the Matrix A, by finding a basis for its row space, where

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 0 \\ 2 & 0 & -4 \\ 1 & 1 & 3 \end{bmatrix}$$

21. (a) If D is any alternating n-linear function on $K^{n\times n}$, then prove that for each n x n matrix A, D(A)=(det A)D(I).

    (b) If A and B are two n x n matrices over K, then show that det (AB) = (det A) (det B).

22.
    1. Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Prove that T is diagonalizable if and only if the minimal polynomial for T has the form $p = (x - c_1)(x - c_2)\cdots(x - c_k)$ where $c_1, c_2, \cdots, c_k$ are distinct elements of F.
    2. *Every matrix A such that $A^2 = A$ is similar to a diagonal matrix*

(2×5=10 weightage)