QP CODE: 23004833

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Reg No : ....

Name : ....

MSc DEGREE (CSS) EXAMINATION , JULY 2023

Second Semester

CORE - ME010205 - MEASURE AND INTEGRATION

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

4ACB7898

Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define Borel sets. Show that every interval is a Borel set.
2. State and prove the excision property of measurable sets.
3. Define Cantor-Lebesgue function.
4. Define a simple function and its canonical form. Also define pointwise convergence of a sequence of functions. State the simple approximation Theorem.
5. Verify by an example that pointwise convergence of a sequence $\{f_n\}$ of bounded Lebesgue measureable functions on a set of finite measure E, is not sufficient for passage of limit under integral sign
6. State and prove monotone convergence theorem for nonnegative Lebesgue measurable functions.
7. Define $\sigma-$finite measure space. Prove that the Lebesgue measure on R is $\sigma-$ finite.
8. If ν is a signed measure on a measurable space $(X, \mathcal{M})$, show that for each $E \in \mathcal{M}$, $-\nu^-(E) \leq \nu(E) \leq \nu^+(E)$ and $|\nu(E)| \leq |\nu|(E)$, where ν^+ and $\nu^-$are the Jordan decompositions of ν.
9. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a nonnegative measurable function on X for which $\int_X f\, d\mu < \infty$. Then prove that f is finite a.e. on X and $\{x \in X / f(x) > 0\}$is σ finite.
10. State Tonelli's Theorem.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. If $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, then prove that $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(E_k)$

12. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of Lebesgue measurable sets. Prove that for any set A, $m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} E_k\right)\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$

13. Define Lebesgue measurability of a function. Prove that a function f on a Lebesgue measurable set E is Lebesgue measurable if and only if the inverse image under f of every open set is Lebesgue measurable.

14. Define Riemann Integrablity of f over [a,b] . Show that the pointwise limit of sequence of Riemann integrable functions need not be Riemann integrable

15. Prove that finite union of measurable sets is measurable.

16. Let E be a measurable subset of X and f an extended real valued function on X. Show that f is measurable if and only if its restrictions to E and $X \sim E$ are measurable.

17.
    1. *Let $(X, \mathcal{M}, \mu)$ be a measure space and ψ be nonnegative simple functions on X. If* $X_0 \subseteq X$ is measurable and $\mu(X - X_0) = 0$, then prove that $\int_X \psi \, d\mu = \int_{X_0} \psi \, d\mu$
    2. Let $(X, \mathcal{M}, \mu)$ be a measure space, ϕ and ψ be nonnegative simple functions on X . If $\psi \leq \phi$ a.e on X then $\int_X \psi \, d\mu = \int_X \phi \, d\mu$

18. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of functions on X that is both uniformly integrable and tight over X. If $\{f_n\} \to f$ pointwise a.e. on X and the function f is integrable over X, prove that $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19.
    1. Let E be a bounded measurable set of real numbers. Suppose there is a bounded, countably infinite set Λ of real numbers for which the collection of translates of $\{\lambda + E\}_{\lambda \in \Lambda}$ is disjoint. Then prove that $m(E) = 0$.
    2. State and Prove Vitali's theorem.

20. Prove that Lebesgue integration of bounded Lebesgue measurable functions on sets of finite measure satisfies the properties of Linearity, Monotonicity and Additivity over domains of integration.

21. Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Prove that there exists a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$ Also prove that the pair $\{A, B\}$ is unique except for null sets.

22. State and prove Radon Nikodym Theorem.

(2×5=10 weightage)