QP CODE: 23004829

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Reg No : ....

Name : ....

MSc DEGREE (CSS) EXAMINATION , JULY 2023

Second Semester

CORE - ME010201 - ADVANCED ABSTRACT ALGEBRA

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 Admission Onwards

F3683C40

Time: 3 Hours Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. *Find* $irr(\sqrt{1+\sqrt{3}}, \mathbb{Q})$
2. *If* γ *is constructible and* $\gamma \notin \mathbb{Q}$, *then prove that* $[\mathbb{Q}(\gamma) : \mathbb{Q}] = 2^r$ for *some integer* $r \geq 0$.
3. Check whether the function ν for the integral domain $\mathbb{Z}[x]$ given by $\nu(f(x)) =$ (degree of $f(x)$) for $f(x) \in \mathbb{Z}[x], f(x) \neq 0$ is a Euclidean norm.
4. State Euclidean algorithm.
5. Define a multiplicative norm in $\mathbb{Z}[i]$, which is a Euclidean norm also.
6. *Let* α *be algebraic of degree* n *over a field* F. *Prove that there are at most* n *different isomorphisms of* $F(\alpha)$ *onto a subfield of* $\overline{F}$ *and leaving* F *fixed.*
7. *If* $E \leq \overline{F}$ *is an algebraic extension of a field* F *and* $\alpha, \beta \in E$ *are conjugate over* F, *then prove that the conjugation isomorphism* $\psi_{\alpha,\beta} : F(\alpha) \to F(\beta)$ *can be extended to an isomorphism of* E *onto a subfield of* $\overline{F}$.
8. *Define the index of* E *over* F *where* E *be a finite extension of a field* F. *Illustrate the definition with an example.*
9. Let K be a finite normal extension of a field F and let E be an extension of F, where $F \leq E \leq K \leq \overline{F}$. Prove that K is a finite normal extension of E.
10. *Define the* nth *cyclotomic extension of a field* F. *Give an example.*

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. *Let* E *be an algebraic extension of a field* F *. Then prove that there exist a finite number of elements* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *in* E *such that E=* $F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *if and only if* E *is a finite dimensional vector space over* F

12. *Prove that a field is algebraically closed if and only if every non constant polynomial in F[x] factors in F[x] into linear factors. Also show that an algebraically closed field has no proper algebraic extensions.*

13. Let D be a PID. Prove that every element that is neither 0 nor a unit in D is a product of irreducibles.

14. If D is a UFD, then prove that for every nonconstant f(x) in D[x], f(x) = (c)g(x), where c belongs to D and g(x) in D[x] is primitive. Also prove that the element c is unique upto a unit factor in D and g(x) is unique upto a unit factor in D.

15. *Let* F *be a subfield of a field* E. *Prove that the set* $G(E/F)$ *of all automorphisms of* E *leaving* F *fixed forms a subgroup of the group of all automorphisms of* E. *Also prove that* $F \leq E_{G(E/F)}$.

16. *Show that if* $[E : F] = 2$, *then* E *is a splitting field over* F.

17. *Let* F *be a field and* $f(x)$ *be an irreducible polynomial in* $F[x]$. *Prove that all zeros of* $f(x)$ *in* $\overline{F}$ *have the same multiplicity.*

18. Let K be a finite extension of degree n of a finite field F of p^r elements. Prove that $G(K/F)$ is cyclic of order n and is generated by σ_{p^r}, where for $\alpha \in K$, $\sigma_{p^r}(\alpha) = \alpha^{p^r}$

(6×2=12 weightage)

Part C (Essay Type Questions)

Answer any **two** *questions.*

Weight **5** *each.*

19. a) Prove that a finite field $GF(p^n)$ of p^n elements exists for every prime power p^n.
   b) If E *and* E' *are fields of order* p^n, *then prove that* E≃ E '.

20. a) If D is a UFD, then prove that a product of two primitive polynomials in D[x] is again primitive.
   b) If D is a UFD, then prove that D[x] is a UFD.

21. *Define splitting field over a field* F. *Prove that a field* E, $F \leq E \leq \overline{F}$ *is a splitting field over* F *if and only if every automorphism of* $\overline{F}$ *leaving* F *fixed maps* E *onto itself and thus induces an automorphism of* E *leaving* F *fixed.*

22. Prove the following.
   a) Every field of characteristic zero is perfect.
   b) Every finite field is perfect.

(2×5=10 weightage)