QP CODE: 23004010

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Reg No : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, JUNE 2023

Fourth Semester

Elective - ME800401 - DIFFERENTIAL GEOMETRY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

BBCCA011

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Sketch the level sets $f^{-1}(0)$ and typical values $\nabla f(p)$ of the vector field ∇f for the function $f(x_1, x_2) = x_1^2 - x_2^2$ when $p \in f^{-1}(0)$.

2. Define an n-surface S and the tangent space S_p at a point $p \in S$. Give an example.

3. What is Gauss Map? Sketch the Gauss Map for 1-surface in $\mathbb{R}^2$

4. Describe the spherical image, when $n = 2$, of the cone $-x_1^2 + x_2^2 + x_3^2 + \ldots + x_{n+1}^2 = 0, x_1 > 0$ oriented by its unit normal vector field.

5. If $\mathbf{X}$ and $\mathbf{Y}$ are parallel vector fields along a parametrized curve α, then show that $\mathbf{X} + \mathbf{Y}$ and $c\mathbf{X}$ are parallel, for all $c \in \mathbb{R}$

6. Define the derivative of a smooth vectorfield $\mathbf{X}$ on an open set U in $\mathbb{R}^{n+1}$ with respect to a vector $\mathbf{v} \in \mathbb{R}_p^{n+1}$, $p \in U$. Show that $\nabla_{\mathbf{v}}(f\mathbf{X}) = (\nabla_{\mathbf{v}} f)\mathbf{X}(p) + f(p)(\nabla_{\mathbf{v}}\mathbf{X})$ where $f : U \to \mathbb{R}$ is a smooth function.

7. Explain a) Circle of curvature.

   b) Center of curvature.

   c) Radius of curvature of a plane curve at the point p.

8. Let U be an open set in $\mathbb{R}^{n+1}$ and $f : U \to \mathbb{R}$ is smooth. Then for any parametrized curve $\alpha : I \to U$ evaluate $\int_\alpha df$.

9. Prove that a parametrized 1-surface is simply a regular parametrized curve.

10. Let $\varphi : U \to \mathbb{R}^{n+k}$ be any smooth map, U open in $\mathbb{R}^n$.

    a) Define vector field along φ.

    b) Let $p \in U$ and $\mathbf{v} \in \mathbb{R}_p^n$. Define the derivative of a smooth vector field $\mathbf{X}$ with respect to $\mathbf{v}$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. How can you obtain level sets from graph and graph from level sets? Explain the process using the level sets and graphs of the function $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \ldots + x_{n+1}^2$ for $n = 0$ and $n = 1$.

12. If S is a connected n-surface in $\mathbb{R}^{n+1}$ and $g : S \to \mathbb{R}$ is smooth and takes on only the values +1 and -1, then prove that g is a constant.

13. For each pair of orthogonal unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^3$ and $a \in \mathbb{R}$, verify the great circle $\alpha(t) = (\cos at)e_1 + (\sin at)e_2$ is a geodesic in the 2-sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$.

14. Let $\alpha : [0, \pi] \to S^2$ be the half great circle in S^2, running from the north pole $p = (0,\ 0,\ 1)$ to the south pole $q = (0,\ 0, -1)$. defined by $\alpha(t) = (sint,\ 0,\ cost)$. Show that, for $\mathbf{v} = (p,\ v_1,\ v_2,\ 0) \in S_p^2$, $P_\alpha(\mathbf{v}) = (q,\ -v_1,\ v_2,\ 0)$

15. a) What do you mean by reparametrization of a parametrized curve.

    b) Are local parametrizations of plane curves unique upto reparametrization?Justify your answer.

16. Define length of a connected oriented plane curve. Find the length of the connected oriented plane curve $f^{-1}(c)$ oriented by $\frac{\nabla f}{\|\nabla f\|}$, where $f : U \to \mathbb{R}$ is given by $f(x_1, x_2) = 5x_1 + 12x_2, U = \{(x_1, x_2) : x_1^2 + x_2^2 < 169\}, c = 0$.

17. *Let V be a finite dimensional vector space with dot product and let $L: V \to V$ be a self adjoint linear transformation on V. Let $S = \{v \in V: v \cdot v = 1\}$ and define $f: S \to \mathbb{R}$ by $f(v) = L(v) \cdot v$. Prove that if v_0 is an eigen vector of L, then f is stationary at $v_0 \in S$.*

18. *Find the normal curvature $k(\mathbf{v})$ for each tangent direction $\mathbf{v}$, the principal curvatures and principal curvature directions, and the Gauss-Kronecker and mean curvatures, at the point $p = (1, 0, \ldots, 0)$ of the n-surface $x_1 + x_2 + \ldots + x_{n+1} = 1$ oriented by the outward normal vector field.*

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. a) Let $\mathbf{X}$ be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Then prove that there exists an open interval I containing 0 and an integral curve $\alpha : I \to U$ of $\mathbf{X}$ such that

    $(i)\alpha(0) = p$

    (ii) If $\beta : \tilde{I} \to U$ is any other integral curve of $\mathbf{X}$ with $\beta(0) = p$ then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$

    b) Given the vector field$(p) = (p.\mathbf{X}(p))$ where $\mathbf{X}(p) = (-p)$. Then find the integral curve through an arbitrary point $(1, 1)$.

20. Show that each maximal geodesic on the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$ is either a vertical line, a horizontal circle, a helix or a constant.

21. a) Let S be an n-surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field $\mathbf{N}$. Let $p \in S$ and $\mathbf{v} \in S_p$. Prove that $\ddot{\alpha}(t_0) \cdot \mathbf{N}(p) = L_p(\mathbf{v}) \cdot \mathbf{v}$ for every parametrized curve $\alpha : I \to S$ with $\dot{\alpha}(t_0) = \mathbf{v}$ for some $t_0 \in I$.

b) Show that if S is an n-surface and $\mathbf{N}$ is a unit normal vector field on S, then the Weingarten map of S oriented by $-\mathbf{N}$ is the negative of the Weingarten map of S oriented by $\mathbf{N}$.

22. *a) Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $p \in S$. Let $\mathbf{Z}$ be any non-zero normal vector field on S such that $\mathbf{N} = \mathbf{Z}/\|\mathbf{Z}\|$ and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for S_p. Prove that*

$$\mathbf{K}(p) = \frac{(-1)^n \det \begin{pmatrix} \nabla_{\mathbf{v}_1}\mathbf{Z} \\ \vdots \\ \nabla_{\mathbf{v}_n}\mathbf{Z} \\ \mathbf{Z}(p) \end{pmatrix}}{\|\mathbf{Z}(p)\|^n \det \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \\ \mathbf{Z}(p) \end{pmatrix}}$$

b) Let S be the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$ oriented by the outward normal vector field. Find the Gauss - Kronecker curvature of S.

(2×5=10 weightage)