21.07.23

**QP CODE: 23004009**

23004009

**Reg No** : ....................

**Name** : ....................

# M Sc DEGREE (CSS) EXAMINATION, JUNE 2023

## Fourth Semester

## Core - ME010402 - ANALYTIC NUMBER THEORY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

B16B5F44

Time: 3 Hours

Weightage: 30

### Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Prove the following (a) $\phi(p^{\alpha}) = p^{\alpha} - p^{\alpha-1}$ for prime $p$ and $\alpha \geq 1$.
   (b)$\phi(mn) = \phi(m)\phi(n)(\frac{d}{\phi(d)})$, where $d = (m, n)$.
2. Define the divisor function $\sigma_{\alpha}(n)$. Show that it is multiplicative.
3. Define average order, big oh notation and asymptotic equality of arithmetical functions.
4. State Abel's identity and deduce Euler's summation formula from it.
5. Let $\{a(n)\}$ be a nonnegative sequence such that $\sum_{n\leq x} a(n)[\frac{x}{n}] = x \log x + O(x)$ for all $x \geq 1$. Then prove that $\forall\, x \geq 1$, we have $\sum_{n\leq x} \frac{a(n)}{n} = \log x + O(1)$.
6. (a)If $a \equiv b (mod\, m)$ and $a \equiv b (mod n)$ where $(m\ , n) = 1$ then prove that $a \equiv b (mod\, mn)$.
   (b) If $a \equiv b (mod\, m)$ and if $0 \leq |b - a| < m$ then prove that $a = b$.
7. Prove that if a prime $p$ does not divide $a$ then $a^{p-1} \equiv 1 (mod\, p)$.
8. A prime $p$ satisfies $(p-1)! \equiv -1 (mod\, p)$. Is the converse true.
9. For every odd prime p, Prove that $(-1|p) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 (mod 4) \\ -1 & \text{if } p \equiv 3 (mod 4). \end{cases}$
10. (a) Define the exponent of $a$ modulo $m$.
    (b) Let $m \geq 1$ and $(a, m) = 1$. Then prove that $a^k \equiv a^h (modm)$ if and only if $k \equiv h (mod\, m)$, where $f = exp_m(a)$.

(8×1=8 weightage)

### Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Show that two lattice points $(a, b)$ and $(m, n)$ are mutually visible if and only if $a - m$ and $b - n$ are relatively prime.

12. For $x \geq 2$ , prove that $\sum_{p\leq x}[\frac{x}{p}]log\, p = x\, log\, x + O(x)$, where the sum is extended over all primes $\leq x$.

13. If $n \geq 1$ ,show that $\frac{1}{6}n\, log\, n < P_n < 12(n\, log\, n + n\, log\frac{12}{e})$, where $P_n$ denotes the $n^{th}$prime .

14. Find a constant $A$such that $\sum_{p\leq x}\frac{1}{p} = log\, log\, x + A + O(\frac{1}{log\, x}), \forall\, x \geq 2.$

15. Assume (a,m)= d and suppose that $d|b$ . Then prove that the linear congruence $ax \equiv b(mod m)$ has exactly d solutions modulo m given by $t, t+\frac{m}{d}, t+2\frac{m}{d}, \ldots . t+(d-1)\frac{m}{d}$ where t is a solution unique modulo $\frac{m}{d}$ of the linear congruence $\frac{a}{d}x \equiv \frac{b}{d}(mod\ \frac{m}{d})$.

16. Assume $m_1, \ldots, m_r$ are positive integers, relatively prime in pairs. Let $b_1, \ldots, b_r$ be arbitrary integers. Then prove that the system of congruences $x \equiv b_1(mod m_1), \ldots, x \equiv b_r(mod m_r)$ has exactly one solution modulo $m_1, \ldots, m_r$.

17. Define Legendre's symbol $(n|p)$. Prove that it is a completely multiplicative function.

18. Let p be an odd prime. Prove that there are exactly $\phi(p-1)$ primitive roots mod p.

(6×2=12 weightage)

**Part C (Essay Type Questions)**

*Answer any **two** questions.*

*Weight **5** each.*

19. (a) State and prove the theorem which give the recursion formulas for $f^{-1}$, where $f$ is an arithmetical function with $f(1) \neq 0$.

(b) Prove that if both $g$ and $f * g$ are multiplicative then $f$ is multiplicative.

20. Show that the following relations are logically equivalent.

$(a)\ \lim_{x\to\infty} \frac{\pi(x)logx}{x} = 1.$

$(b)\ \lim_{x\to\infty} \frac{\pi(x)\ log\ \pi(x)}{x} = 1.$

$(c)\ \lim_{n\to\infty} \frac{P_n}{nlogn} = 1, where\ P_n\ denotes\ the\ n^{th} prime.$

21. (a) Let f be a polynomial with integer coefficents, let $m_1$, $m_2$, ..., $m_r$ be positive integer relatively prime in pairs, and let m = $m_1 m_2 \ldots m_r$.Prove that the congruence $f(x) \equiv 0(mod m)$ has a solution if and only each of the congruence $f(x) \equiv 0(mod m_i)$ , i=1,,2,...,r , has a solution.

Also show that if v(m) and v($m_i$) denote the number of solutions of $f(x) \equiv 0(mod m)$ and $f(x) \equiv 0(mod m_i)$ , i=1,,2,...,r respectively, then v(m)= v($m_1$)v($m_2$)...v($m_r$).

(b) Prove that for a given any integer k > 0 there exists a lattice point (a , b) such that none of the lattice points ( a+r, b+s), 0<r, s < k is visible from the orgin.

22. Derive a formula for $(p|q)(q|p)$, where p and q are distinct odd primes.

(2×5=10 weightage)