QP CODE: 23004008

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Reg No : ....

Name : ....

M Sc DEGREE (CSS) EXAMINATION, JUNE 2023

Fourth Semester

Core - ME010401 - SPECTRAL THEORY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

CA43BFC1

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define Canonical mapping of a normed space X into X''. Prove that Canoical mapping is linear.
2. If (x_n) and (y_n) are sequences in the same normed space X, show that $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ implies $x_n + y_n \xrightarrow{w} x + y$.
3. Define contraction on a metric space. State Banach fixed point Theorem.
4. Define characteristic equation and eigenvalues of an $n \times n$ matrix.
5. Let X be a complex Banach space, $T \in B(X, X)$ and $\lambda, \mu \in \rho(T)$. Then prove that $R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda$.
6. Define a commutative algebra with identity. Give an example.
7. Define totally bounded sets. Is a totally bounded set of a metric space is bounded?
8. Let T and S be linear operators in a normed space X. If T is compact and S is bounded prove that TS is compact.
9. Let T_1, T_2, T be bounded self-adjoint linear operators on a complex Hilbert space H and $\alpha \geq 0$. If $T_1 \leq T_2$, prove that $T_1 + T \leq T_2 + T$ and $\alpha T_1 \leq \alpha T_2$.
10. Let P_1 and P_2 be projections defined on a Hilbert space H and let $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$. If the difference $P = P_2 - P_1$ is a projection, prove that $P_2 P_1 = P_1 P_2 = P_1$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Prove that a sequence (f_n) of bounded linear functionals on a Banch space X is weak* convergent if and only if

(A) The sequence $(\|f_n\|)$ is bounded.

(B) The sequence $(f_n x)$ is cauchy for every x in a total subset M of X.

12. Let $T : \mathcal{D}(T) \to Y$ be a bounded linear operator with domain $\mathcal{D}(T) \subset X$, where X and Y are normed spaces. Prove that

    (a) If $\mathcal{D}(T)$ is a closed subset of X, then T is closed.

    (b) If T is closed and Y is complete, then $\mathcal{D}(T)$ is a closed subset of X.

13. Let $T : X \to X$ be a bounded linear operator on a complex normed space X and $\lambda \in \rho(T)$. Then prove that the resolvent operator $R_\lambda(T)$ is defined on the whole space X.

14. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.

15. Let A be a complex Banach algebra with identity. Then show that the set G of all invertible elements of A is an open subset of A.

16. Let X and Y be normed space and $T : X \to Y$ a linear operator. State and prove the characterisation theorem for T to become a compact linear operator.

17. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space $H \neq \{0\}$. Prove that $m = \inf_{\|x\|=1} \langle Tx, x\rangle$ and $M = \sup_{\|x\|=1} \langle Tx, x\rangle$ are spectral values of T.

18. Let P_1 and P_2 are projections on a Hilbert space H. Prove that the product $P = P_1 P_2$ is a projection on H if and only if P_1 and P_2 commute. Also prove that P projects H onto $Y = Y_1 \cap Y_2$, where $Y_j = P_j(H)$, $(j = 1, 2)$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. State and prove Open Mapping Theorem

20. Let $T \in B(X, X)$, where X is a complex Banach space. Then:

    (a) If $\|T\| < 1$, prove that $(I - T)^{-1}$ exists as a bounded linear operator on the whole space X and $I - T)^{-1} = \sum_{j=0}^{\infty} I + T + T^2 + \cdots$

    (b) Prove that the spectrum of T is closed.

21. Let $T : X \to X$ be a compact linear operator on a normed space X, and $\lambda \neq 0$. Prove that there exists a smallest integer r such that from $n = r$ onwards the null spaces $\mathcal{N}(T_\lambda^n)$ are all equal and the ranges $T_\lambda^n(X)$are all equal and if $r > 0$, the following inclusions are proper.

    $\mathcal{N}(T_\lambda^0) \subset \mathcal{N}(T_\lambda) \subset \mathcal{N}(T_\lambda^2) \subset \cdots \subset \mathcal{N}(T_\lambda^r)$ and

    $T_\lambda^0(X) \supset T_\lambda(X) \supset T_\lambda^2(X) \supset \cdots \supset T_\lambda^r(X)$.

22.
    1. Prove that eigen vectors corresponding to different eigenvalues of a bounded self-adjoint linear operator on a complex Hilbert space are orthogonal.
    2. State and prove a characterization of the resolvent set of a bounded self-adjoint linear operator on a complex Hilbert space.

(2×5=10 weightage)