24000223

24000223

Reg. No...............................

Name...................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, JANUARY 2024

Third Semester

Faculty of Science

Branch I—(A) Mathematics

MT 03 C 13—DIFFERENTIAL GEOMETRY

(2018 Admissions–Supplementary/2017,2016 and 2015 Admissions–Mercy Chance)

Time : Three Hours | Maximum Weight : 30

Part A

Answer any **five** *out of eight questions.*
Each question has weight 1.

1. Find and sketch the gradient field of $f(x_1, x_2) = x_1 + x_2$.
2. Show that the gradient of a smooth function of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p.
3. Define the terms Gauss map and shperical image.
4. Define the covariant derivative of a smooth vector field.
5. What is the geometric meaning of the Weingarten map L_p.
6. Define the terms circle of curvature of a plane curve.
7. Explain the terms principal curvature and principle curvature directions of an oriented n surface S at a point $p \in S$.
8. What is diffeomorphism ?

$(5 \times 1 = 5)$

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$.
10. Find the integral curve through $p = (1, 1)$ for the vector field $X(x_1, x_2) = (x_2, x_1)$.

Turn over

11. Describe the spherical image when $n = 2$ of the n-surface oriented by $\nabla f / \| \nabla f \|$, where f is the function defined by the left hand side of the equation, the cylinder $x_2^2 + x_3^2 + \ldots x_{n+1}^2 = 1$.

12. Compute $\nabla_v f$ where $f(x_1, x_2) = 2x_1^2 + 3x_2^2, v = (1,0,2,1)$.

13. Find the velocity, acceleration and speed of the curve $d(t) = (\cos t, \sin t)$.

14. Find the length of the parametrized curve $\alpha : I \to R^{n+1}$, where $\alpha(t) = (t^2, t^3), I = [0,2], n = 1$.

15. Find the normal curvature $k(v)$ for each tangent direction v for the surface $x_1 + x_2 + \ldots + x_{n+1} = 1$ at $p = (1,0,\ldots\ldots0)$.

16. Define the differential of a smooth map $\phi : U \to R^m$, where U is an open set in R^n.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. (a) Sketch the cylinder $f^{-1}(0)$, where $f(x_1, x_2, x_3) = x_1 - x_2^2$.

(b) Let U be an open set in R^{n+1} and let $f : U \to R$ be smooth. Let $p \in U$ be a regular point of f and let $c = f(p)$. Show that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

(c) Sketch the level sets $f^{-1}(-1), f^{-1}(0)$ and $f^{-1}(1)$ for $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_n^2 - \ldots x_{n+1}^2$ for $n = 2$.

18. (a) Let S be an n-surface in $R^{n+1}, \alpha : I \to S$ be a parametrized curve in S, to $\in$ I and $v \in S_{\alpha}(\text{to})$. Show that there is a unique vector field V, tangent to S along α, which is parallel and has V (to) = v.

(b) Define Levi-Civita parallelism and explain its properties.

19. (a) Define the Weingarten map and show that it is self adjoint.

(b) Compute $\nabla v X$, where $v \in R_p^{n+1}, p \in R^{n+1}$, and X are given by $X(x_1, x_2) = (x_1, x_2, x_1 \cdot x_2, x_2^2)$, $v = (1,0,0,1)$ and $n = 1$.

24000223

20. Let C be an oriented plane curve. Prove that there exists a global parametrization of C if and only if C is connected.

21. (a) Find the Gaussian curvature $k : S \to R$ where S is the cone $x_1^2 + x_2^2 - x_3^2 = 0, x_3 > 0$.

(b) Find the Gauss-Kronecker curvature of the parametrized 3-surface ϕ, where

$\phi(x, y, z) = (x, y, z, x^2 + y^2 + z^2)$.

22. (a) State and prove inverse function theorem.

(b) Let S be a compact connected oriented n surface in R^{n+1}. Whose Gauss–Kronecker curvature is nowhere zero. Show that the Gauss map $N = S \to S^n$ is a diffeomorphism.

(3 × 5 = 15)