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Reg. No..............................

Name..................................

M.Sc. DEGREE (C.S.S.) EXAMINATION, APRIL 2019

Fourth Semester

Faculty of Science

Branch I (A) : Mathematics

MTO 4C 16—SPECTRAL THEORY

[Programme–Core—Common for all]

(2012 Admission onwards)

Time : Three Hours | Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question has weight 1.

1. Define weak convergence in a normed space. Let (x_n) and (y_n) be two sequence in a normed space X such that $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$. Then prove that $x_n + y_n \xrightarrow{w} x + y$.

2. Let $X = C[0,1]$ and define $T : \mathscr{D}(T) \to X$ by $Tx = x'$, where the prime denotes differentiation and $\mathscr{D}(T)$ is the subspace of functions $x \in X$ which have a continuous derivative. Prove that T is not bounded but is closed.

3. Prove that similar matrices have the same eigenvalues.

4. Let $S, T \in B(X, X)$, show that for any $\lambda \in \rho(S) \cap \rho(T)$ $R_\lambda(S) - R_\lambda(T) = R_\lambda(S)\ (T - S)\, R_\lambda(T)$.

5. Define compact linear operator. Let X be a normed space with dim $X = \infty$. Prove that the identity operator $I : X \to X$ is not compact.

6. Consider the space l^2. Let $T : l^2 \to l^2$ defined by $T(\xi_1, \xi_2, \ldots) = \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \ldots\ldots, \frac{\xi_n}{n}, 0, 0, \ldots.\right)$. Prove that T is compact.

Turn over

7. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space. Then prove that all eigenvectors corresponding to different eigenvalues of T are orthogonal.

8. Let $P : H \to H$ be a bounded linear operator on a Hilbert space H. Suppose P is self-adjoint indempotent. Prove that P is a projection.

(5 × 1 = 5)

Part B

Answer any **five** *questions.*
Each question has weight 2.

9. Let (x_n) be a sequence in a normed space X. Prove that :

   (i) Strong convergence implies weak convergence with the same limit.

   (ii) If $\dim X < \infty$, then weak convergence implies strong convergence.

10. State and prove closed graph theorem.

11. Define closed linear operator. Prove that the graph $\mathscr{G}(T)$ of a linear operator $T : X \to Y$ is a subspace of $X \times Y$.

12. Prove that the resolvent set $\rho(T)$ of a bounded linear operator T on a complex Banach space is open.

13. Let A be a complex Banach algebra with identity e. Let $x \in A$ and $\| x \| < 1$. Prove that $e - x$ is invertible and $(e - x)^{-1} = e + \sum_{j=1}^{\infty} x^j$.

14. Let A be a complex Banach algebra with identity e. Then for any $x \in A$ prove that $\sigma(x)$ is compact.

15. Prove that a compact linear operator $T : X \to Y$ from a normed space X into a Banch space has a compact linear extension $\tilde{T} : \hat{X} \to Y$, where $\hat{X}$ is the completion of X.

16. let H be a complex Hilbert space and $T : H \to H$ be a bounded self-adjoint linear operator. Then prove that $m = \inf_{\| x \| = 0} \langle Tx, x \rangle$ and $M = \sup_{\| x \| = 1} \langle Tx, x \rangle$ are spectral values of T.

(5 × 2 = 10)

Part C

Answer any **three** *questions.*
Each question has weight 5.

17. State and prove open mapping theorem.

18. Let X be a complex Banach space and $T \in B(X, X)$. Let $r_\sigma(T)$ be spectral radius of T. Then prove that $r_\sigma(T) = \lim_{n \to \infty} \sqrt[n]{\| T^n \|}$.

19. Let $T : X \to Y$ be a compact linear operator. Prove that its adjoint operator $T^x : Y' \to X'$ is a compact linear opeartor, where X and Y are normed space and X' and Y' are dual spaces of X and Y.

20. (a) Let (T_n) be a sequence of compact linear operators from a normed space X into a Banach space Y. If (T_n) is uniformly operator convergent to T, then prove that T is compact.

    (b) Let $T : l^2 \to l^2$ defined by $T(\xi_1, \xi_2, \ldots.) = \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \ldots., \frac{\xi_n}{n}, \ldots\right)$. Prove that T is a compact linear operator.

    (c) Let X and Y be normed spaces and $T : X \to Y$ a compact linear operator. Suppose that (x_n) in X is weakly convergent, say $x_n w x$ then prove that (Tx_n) is strongly convergent in Y and has the limit $y = T_x$.

21. If two bounded self-adjoint linear operators S and T on a Hilbert space H are positive and commute then prove that their product ST is positive.

22. Let P_1 and P_2 be two projections on a Hilbert space H. Then prove that :

    (i) $P = P_2 - P_1$ is a projection on H if and only if $Y_1 \subset Y_2$ where $Y_i = P_i(H), i = 1,2$.

    (ii) If $P = P_2 - P_1$ is a projection, P projects H onto Y, where Y is orthogonal complement of Y_1 in Y_2.

    (iii) $P_1 + P_2 - P_1P_2$ is a projection if $P_1P_2 = P_2P_1$.

(3 × 5 = 15)