

QP CODE: 24018058



Reg No : .....

Name : .....

**M Sc DEGREE (CSS) EXAMINATION, APRIL 2024**

**Fourth Semester**

**Core - ME010402 - ANALYTIC NUMBER THEORY**

M Sc MATHEMATICS, M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

F8DFAACE

Time: 3 Hours

Weightage: 30

Instructions (Applicable for **Private Registration, 2020 Admission Onwards**) : This question paper contains two sections. Answer section I questions in the answer book provided. Section II Internal examination questions must be answered in the question paper itself. Follow the detailed instructions given under section II.

**SECTION I**

**Part A (Short Answer Questions)**

Answer any **eight** questions.

Weight **1** each.

1. Prove that the *Möbius* function is multiplicative but not completely multiplicative.
2. If  $\alpha$  has a Dirichlet inverse  $\alpha^{-1}$ , then prove that the equation  $G(x) = \sum_{n \leq x} \alpha(n) F\left(\frac{x}{n}\right)$  implies  $F(x) = \sum_{n \leq x} \alpha^{-1}(n) G\left(\frac{x}{n}\right)$  and conversely.
3. Write an asymptotic formula for  $\sum_{n \leq x} d(n)$  and hence find its average order.
4. State any four relations which are logically equivalent to the prime number theorem.
5. State Shapiro's Taubarian Theorem.
6.
  - (a) If  $c > 0$  then prove that  $a \equiv b \pmod{m}$  if and only if  $ac \equiv bc \pmod{mc}$ .
  - (b) Assume  $a \equiv b \pmod{m}$ . If  $d \mid m$  and  $d \mid a$  then prove that  $d \mid b$ .
7. If  $(a, m) = 1$  then show that the solution of the linear congruence  $ax \equiv b \pmod{m}$  is given by  $x \equiv ba^{\phi(m)-1} \pmod{m}$ .

8. State and prove Wilson's theorem.
9. Prove that  $(-1|p) = 1$  if  $p \equiv 1 \pmod{4}$ . Also write a formula for  $(2|p)$  when  $p$  is an odd prime.
10. (a) Define the exponent of  $a$  modulo  $m$ .  
 (b) Let  $m \geq 1$  and  $(a, m) = 1$ . Prove that the numbers  $1, a, a^2, \dots, a^{f-1}$  are incongruent mod  $m$ , where,  $f = \exp_m(a)$ .

(8×1=8 weightage)

### Part B (Short Essay/Problems)

Answer any **six** questions.

Weight **2** each.

11. (a) Define Dirichlet product of two arithmetical functions  $f$  and  $g$ .  
 (b) Prove that  $f * g = g * f$  and  $(f * g) * k = f * (g * k)$  for any arithmetical functions  $f, g$  and  $h$ .
12. (a) For  $x \geq 1$ , prove that  $\sum_{n \leq x} \Lambda(n) \left[ \frac{x}{n} \right] = \log[x]!$ .  
 (b) State and prove the Legendre's identity.
13. State and prove Abel's identity.
14. Prove that there is a constant  $A$  such that  $\sum_{p \leq x} \frac{1}{p} = \log \log x + A + O\left(\frac{1}{\log x}\right), \forall x \geq 2$ .
15. (a) Assume  $(a, m) = 1$ , show that the linear congruence  $ax \equiv b \pmod{m}$  has exactly one solution.  
 (b) Give an example of a linear congruence having no solution.
16. Prove that the set of lattice points in the plane visible from the origin contains arbitrarily large square gaps.
17. State and prove Euler's criterion.
18. If  $(a, m) = 1$  then prove that  $\exp_m(a^k) = \frac{\exp_m(a)}{(k, f)}$ , where  $f = \exp_m(a)$ .

(6×2=12 weightage)

### Part C (Essay Type Questions)

Answer any **two** questions.

Weight **5** each.

19. (a) For  $x > 1$ , prove that  $\sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$ .  
 (b) Two lattice points  $(a, b)$  and  $(m, n)$  are mutually visible if and only if  $a - m$  and  $b - n$  are relatively prime.
20. State and prove the inequality showing  $\frac{n}{\log n}$  is the correct order of magnitude of  $\pi(n)$ .





21. (a) State and prove Chinese Remainder Theorem .

(b)  $m_1, m_2, \dots, m_r$  are relatively prime in pairs and  $b_1, b_2, \dots, b_r$  are arbitrary integers and let  $a_1, a_2, \dots, a_r$  satisfy  $(a_k, m_k) = 1$  for  $k = 1, 2, \dots, r$ . Prove that the system of congruences  $a_1 x \equiv b_1 \pmod{m_1}, a_2 x \equiv b_2 \pmod{m_2}, \dots, a_r x \equiv b_r \pmod{m_r}$  has exactly one solution modulo  $m_1 m_2 \dots m_r$ .

22. State and prove Gauss' lemma.

(2×5=10 weightage)