QP CODE: 24018060

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Reg No : ....................

Name : ....................

# M Sc DEGREE (CSS) EXAMINATION, APRIL 2024

## Fourth Semester

## Elective - ME800401 - DIFFERENTIAL GEOMETRY

M Sc MATHEMATICS,M Sc MATHEMATICS (SF)

2019 ADMISSION ONWARDS

3B442B56

Time: 3 Hours Weightage: 30

*Instructions: (**Applicable for Private Registration, 2020 Admission Onwards**) This question paper contains two sections. Answer section I questions in the answer book provided. Section II Internal examination questions must be answered in the question paper itself. Follow the detailed instructions given under section II.*

### SECTION I

**Part A (Short Answer Questions)**

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define level set of a function $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^{n+1}$ and explain with an example
2. Define regular point of a smooth function $f : U \to \mathbb{R}$ in an open set $U \subseteq \mathbb{R}^{n+1}$ and find whether $(0, 0)$ is a regular point of $f(x_1, x_2) = x_1^2 + x_2^2$
3. Describe the spherical image, when $n = 1$, of $x_1^2 - x_2^2 - \ldots - x_{n+1}^2 = 4, x_1 > 0$ oriented by $\mathbf{N} = \frac{\nabla f}{\|\nabla f\|}$.
4. Find the velocity, the acceleration and the speed of the parametrized curve $\alpha(t) = (\cos t, \sin t, 2\cos t, 2\sin t)$.
5. Prove that parallel transport is a linear map.
6. Define the derivative of a smooth vectorfield $\mathbf{X}$ on an open set $U$ in $\mathbb{R}^{n+1}$ with respect to a vector $\mathbf{v} \in \mathbb{R}_p^{n+1}$, $p \in U$. Show that $\nabla_\mathbf{v}(\mathbf{X} + \mathbf{Y}) = \nabla_\mathbf{v}\mathbf{X} + \nabla_\mathbf{v}\mathbf{Y}$.
7. Write a formula for finding curvature of a plane curve at the point $p$. Also define curvature of a plane curve.
8. Explain circle of curvature and center of curvature. Also define radius of curvature of a plane curve at the point p.
9. *Define global property. Explain with an example.*

10. Define differential of a smooth map $\varphi : U \to \mathbb{R}^m$, where $U$ is an open set in $\mathbb{R}^n$. Show that the value of the differential does ot depend on the choice of parametrized curve.

(8×1=8 weightage)

## Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let $\mathbf{X}$be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Then prove that there exists an open interval $I$ containing 0 and an integral curve $\alpha : I \to U$ of $\mathbf{X}$ such that

$(i)\alpha(0) = p$

$(ii)$ If $\beta : \tilde{I} \to U$ is any other integral curve of $\mathbf{X}$ with $\beta(0) = p$ then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$

12. Let $S \subset \mathbb{R}^{n+1}$ be a connected $n$-surface in $\mathbb{R}^{n+1}$. Show that there exists on $S$ exactly two unit normal vector fields $\mathbf{N}_1$ and $\mathbf{N}_2$.

13. Verify that great circles are geodesics in the unit 2-sphere.

14. Let $S$ be an $n$ -surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve, and let $\mathbf{X}$ and $\mathbf{Y}$ be vector fields tangent to $S$ along $\alpha$. Show that

a) $(\mathbf{X} + \mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$

b) $(f\mathbf{X})' = f'\mathbf{X} + f\mathbf{X}'$ for all smooth functions $f$ along $\alpha$.

15. Define length $l(\alpha)$ of the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$. Show that length of a parametrized curve is invariant under reparametrization.

16. Let $\eta$ be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by $\eta = -\frac{x_2}{x_1^2+x_2^2}dx_1 + \frac{x_1}{x_1^2+x_2^2}dx_2$. Let $C$ denote the ellipse $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$ oriented by its inward normal. Show that the 1-form $\eta$ is not exact.

17. Find the normal curvature of the 1-sheeted hyperboloid $-x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$ oriented by the inward normal vector field at $p = (0, 0, 1)$ and $\mathbf{v} = (p, 0, 1, 0)$.

18. Find the orientation vector field along the parametrized torus $\varphi$ in $\mathbb{R}^3 : \varphi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$.

(6×2=12 weightage)

## Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Prove the Lagrange Multiplier theorem for an $n$-surface in $\mathbb{R}^{n+1}$by stating the conditions. Hence find the extreme points of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ on the unit circle $x_1^2 + x_2^2 = 1$.

20. Given $S$ is a compact connected oriented $n$-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$, $\forall p \in S$. Is the Gauss map from $S$ to the unit sphere $S^n$onto ? Explain.

21. Prove that the Weingarten map $L_p$ is self-adjoint.

22. *a) Define the second fundamental form of an oriented n-surface in* $\mathbb{R}^{n+1}$ *at a point. When it is said to be positive definite, negative definite and definite?*

*b) Prove that for each compact oriented n-surface S in* $\mathbb{R}^{n+1}$, *there exists a point p such that the second fundamental form at p is definite.*

(2×5=10 weightage)