QP CODE: 24045339

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Reg No : ....

Name : ....

M.Sc DEGREE (CSS) EXAMINATION, DECEMBER 2024

First Semester

CORE - ME010102 - LINEAR ALGEBRA

M.Sc MATHEMATICS,M.Sc MATHEMATICS(SF)

2019 ADMISSION ONWARDS

671A9B59

Time: 3 Hours | Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. *On $\mathbb{R}^n$, define two operations $\alpha \oplus \beta = \alpha - \beta$ and $c \cdot \alpha = -c\alpha$. Which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, \cdot)$?*

2. *Is the vector (3, −1, 0, −1) in the subspace of $\mathbb{R}^4$ spanned by the vectors (2, −1, 3, 2), (−1, 1, 1, −3), and (1, 1, 9, −5)?*

3. Check whether the operator T on $F^{m\times n}$ defined by $T(A) = PAQ$ is linear, where $P \in F^{m\times n}$ and $Q \in F^{n\times n}$ are fixed matrices over the field F.

4. Let V and W be finite dimensional vector spaces over the field F such that $dimV = dimW$ .Prove that V and W are isomorphic.

5. Define the transpose of a linear transformation $T : V \to W$, where V and W are vector spaces over the field F.

6. Check whether the function D on $\mathbb{R}^{3\times 3}$ defined by *$D(A) = A_{13}A_{22}A_{32} + 5A_{12}A_{22}A_{32}$ is 3-linear.*

7. *Find the determinant of the Vandermonde matrix* $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$.

8. *Write any 4 properties of the determinant function on $K^{n \times n}$, where K is a commutative ring with identity.*

9. Find the characteristic values, if any, of the linear operator T on $\mathbb{R}^2$ which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

10. *Suppose that $T\alpha = c\alpha$. If f is any polynomial, then prove that $f(T)\alpha = f(c)\alpha$*

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. *Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, \ldots, \beta_m$. Then show that any independent set of vectors in V is finite and contains no more than m elements.*

12. *Show that the vectors $\alpha_1 = (-1, 0, 0)$, $\alpha_2 = (4, 2, 0)$, $\alpha_3 = (5, 3, -8)$ form a basis for $\mathbb{R}^3$. Find the coordinates of the vector (1, 2, 1) in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$.*

13. Let V be a finite dimensional vector space over the field F and $B = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ and $B' = \{\alpha_1', \alpha_2', \alpha_3', \ldots, \alpha_n'\}$ are ordered bases for V. If T is a linear operator on V prove that there exists an invertible matrix P of order n such that $[T]_{B'} = P^{-1}[T]_B P$.

14. Find the dual basis of the basis $B = \{(1, 0, -1), (1, 1, 1), (2, 2, 0)$ of C^3.

15. If S is any subset of a finite dimensional vector space V and W is the subspace of V spanned by S then prove that $S^{00} = W$

16. *Let K be a commutative ring with identity, and let A and B be n x n matrices over K. Then show that det (AB) = (det A) (det B).*

17. *Find the Characteristic and Minimal polynomials of the matrix* $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

18. Define invariant subspaces. *Let T be a linear operator on $\mathbb{R}^2$, the matrix of which in the standard ordered basis is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find all subspaces of $\mathbb{R}^2$ which are invariant under T.*

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Consider the 5 x 5 matrix A= $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(a) Find an invertible matrix P such that PA is a row-reduced echelon matrix R.

(b) Find a basis for the row space W of A.

(c) Which vectors $(b_1, b_2, b_3, b_4, b_5)$ are in W?

(d) Find the coordinate matrix of each vector $(b_1, b_2, b_3, b_4, b_5)$ in W in the ordered basis chosen in (b).

(e) Write each vector $(b_1, b_2, b_3, b_4, b_5)$ in W as a linear combination of the rows of A.

(f) Give an explicit description of the vector space V of all 5 x 1 column matrices X such that AX = 0.

(g) Find a basis for V.

20. Let V and W be finite dimensional vector spaces over the field F. By exhibiting a basis for the space $L(V,W)$,prove that $L(V,W)$ is finite dimensional and $dimL(V,W) = dimV.dimW$.

21. (a)Let K be a commutative ring with unity, n > 1 and let D be an alternating (n-1)-linear function on $K^{(n-1) \times (n-1)}$. For each j, $1 \leq j \leq n$, show that the function E_j defined by $Ej(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)$ is an alternating n-linear function on n x n matrices A. If D is a determinant function, then show that so is each Ej.

(b)If $K= \mathbb{R}$ and $A= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, find $E_j(A)$ for each j.

22. (a) Let V be a finite dimensional vector space and let W_1 be any subspace of V. Prove that there is a subspace W_2 of V such that $V = W_1 \oplus W_2$

(b) Let $V = W_1 \oplus \cdots \oplus W_k$, prove that there exist k linear operators $E_1, E_2, \cdots, E_k$ on V such that

1. Each E_i is a projection
2. $E_i E_j = 0$ if $i \neq j$
3. $I = E_1 + \cdots + E_k$
4. The range of E_i is W_i

(2×5=10 weightage)