QP CODE: 24045343 24045343

Reg No : ....

Name : ....

M.Sc DEGREE (CSS) EXAMINATION, DECEMBER 2024

First Semester

CORE - ME010104 - REAL ANALYSIS

M.Sc MATHEMATICS,M.Sc MATHEMATICS(SF)

2019 ADMISSION ONWARDS

686C6D58

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. If f is of bounded variation on $[a, b]$, say $\sum \mid \Delta f_k \mid \leq M$ for all partitions of $[a, b]$, then prove that f is bounded on $[a, b]$.
   Also show that $|f(x) \mid \leq \mid f(a) \mid + M, \text{for all } x \in [a, b]$.
2. Define total variation. Prove that $0 \leq V_f < \infty$ and $V_f = 0$ if and only if f is a constant on $[a, b]$.
3. *Define upper and lower Riemann integral.*
4. *Show that* $\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$.
5. State the fundamental theorem of calculus for vector valued functions.
6. Give an example to show that limit processes cannot be interchanged in general for sequence of functions.
7. Is every uniformly convergent sequence of functions pointwise convergent? What about the converse?
8. Under what conditions, a sequence $\{f_n\}$ of continuous functions defined on a compact set K, is convergent uniformly to a continuous function f ?
9. If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then prove that $\{f_n\}$ is uniformly bounded on K.
10. Define the exponential function using power series. State and prove addition formula for the exponential function.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let f be continuous on $[a, b]$. Then prove that f is of bounded variation on $[a, b]$ if, and only if, f can be expressed as the difference of two strictly increasing continuous functions.

12. Let $\mathbf{f} : [a, b] \to \mathbb{R}^n$ and $\mathbf{g} : [c, d] \to \mathbb{R}^n$ be two paths in $\mathbb{R}^n$, each of which is one to one on its domain. Then prove that $\mathbf{f}$ and $\mathbf{g}$ are equivalent if and only if they have the same graph.

13. *If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and if c is a positive constant then show that*
(i) $f \in \mathcal{R}(c\alpha)$ and.
(ii) $\int_a^b f d(c\alpha) = c\int_a^b f d\alpha$.

14. *Suppose φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by $\beta(y)= \alpha(\varphi(y))$, $g(y)= f(\varphi(y))$. Then prove that $g \in \mathcal{R}(\beta)$ and $\int_A^B g d\beta = \int_a^b f d\alpha$.*

15. Suppose $\lim_{n\to\infty} f_n(x) = f(x), \ (x \in E)$ and $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. Then prove that $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

16. Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathscr{R}(\alpha)$on $[a, b]$, for $n = 1, 2, 3, \ldots$ and suppose $f_n \to f$ uniformly on $[a, b]$. Then prove that $f \in \mathscr{R}(\alpha)$ on $[a, b]$.

17. Prove by an example that for a uniformly bounded sequence of continuous functions, there need not exist a pointwise convergent subsequence, even if the domain is compact.

18. If the two series $\sum a_n x^n$ and $\sum b_n x^n$ converges in $S = (-R, R)$, $E = \{x\epsilon S : \sum a_n x^n = \sum b_n x^n\}$ and E has a limit point in S then prove that the given series is identical.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. (i) Explain rectifiable paths and arc lengths with examples.
(ii) Let f and g be complex valued functions defined as follows : $f(t) = e^{2\pi i t}$ if $t \in [0, 1]$ and $g(t) = e^{4\pi i t}$ if $t \in [0, 1]$. Then prove that the length of g is twice that of f.
(iii) Let $f : [a, b] \to \mathbb{R}^n$. Then prove that f is rectifiable if and only if each of the components f_k of f is of bounded variation on $[a, b]$.
Also prove that if f is rectifiable then
$V_k(a, b) \le \Lambda_f(a, b) \le V_1(a, b) + \cdots + V_n(a, b), \qquad k = 1, 2, \ldots, n,$
where $V_k(a, b)$ denotes the total variation of f_k on $[a, b]$

20. *(i) If f is continuous on $[a, b]$ then show that $f \in \mathcal{R}(\alpha)$.*
(ii) If f is monotonic on $[a, b]$ and if α is continuous on $[a, b]$ then prove that $f \in \mathcal{R}(\alpha)$.

21. Prove the existence of a real continuous function on the real line which is nowhere differentiable.

22. If g is a continuous complex function on $[0, 1]$, prove that there exists a sequence of polynomials P_n such that $\lim_{n\to\infty} P_n(x) = g(x)$ uniformly on $[0, 1]$.

(2×5=10 weightage)