QP CODE: 25024354

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Reg No : ....

Name : ....

M.Sc DEGREE (CSS) EXAMINATION, APRIL 2025

Fourth Semester

M Sc MATHEMATICS

ELECTIVE - ME800401 - DIFFERENTIAL GEOMETRY

2019 ADMISSION ONWARDS

71788665

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Find whether the vector field defined by $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1, 0)$ where $U = \mathbb{R}^2$ is complete or not.
2. Define an n-plane. Show that n-plane is an n-surface in $\mathbb{R}^{n+1}$.
3. Describe the spherical image, when $n = 2$, of the cylinder $x_2^2 + x_3^2 + \ldots + x_{n+1}^2 = 1$ oriented by its unit normal vector field.
4. Define derivative of a smooth vector field. Let $\mathbf{X}$ and $\mathbf{Y}$ be smooth vector fields along the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$. Prove $(\mathbf{X} \dot{+} \mathbf{Y}) = \dot{\mathbf{X}} + \dot{\mathbf{Y}}$.
5. Define Covariant derivative of a vector field $\mathbf{X}$. Show that if $\mathbf{X}$ and $\mathbf{Y}$ are smooth vector fields tangent to S along a parametrized curve $\alpha : I \to S$, then $(\mathbf{X} + \mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$.
6. Write a short note on Weingarten map. Explain its geometrical meaning.
7. Define curvature of a plane curve at the point p. Also write a formula for finding curvature.
8. Prove that any oriented plane curve which has global parametrization is connected.
9. *Define global property. Explain with an example.*
10. Define parametrized n-surface in $\mathbb{R}^{n+k} (k \geq 0)$. Give an example. Show that a parametrized 1-surface is simply a regular parametrized curve.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Let $f : U \to \mathbb{R}$ be a smooth function where $U \subseteq \mathbb{R}^{n+1}$ is an open set and $\alpha : I \to U$ be a parametrized curve. Show that $fo\alpha$ is a constant if and only if α is everywhere orthogonal to the gradient of f.

12. Let $S \subset \mathbb{R}^{n+1}$ be a connected n-surface in $\mathbb{R}^{n+1}$. Show that there exists on S exactly two unit normal vector fields $\mathbf{N}_1$ and $\mathbf{N}_2$.

13. Show that if $\alpha : I \to S$ is a geodesic in an n-surface and if $\beta = \alpha \circ h$ is a reparametrization of α where $h : \tilde{I} \to I$ then β is a geodesic in S if and only if there exists $a, b \in \mathbb{R}$ with $a > 0$ such that $h(t) = at + b, \forall t \in \tilde{I}$.

14. For $\theta \in R,$ *let* $\alpha_\theta : [0, \pi] \to S^2$ be the parametrized curve in the unit 2−sphere S^2, from the north pole $p = (0,\ 0,\ 1)$ to the south pole $q = (0,\ 0, -1)$, defined by $\alpha_\theta(t) = (cos\theta\ sint,\ sin\theta\ sint,\ cost)$. Show that, for $\mathbf{v} = (p,\ 1,\ 0,\ 0) \in S_p^2$, $P_{\alpha_\theta}(\mathbf{v}) = -(q,\ cos\ 2\theta,\ sin\ 2\theta,\ 0)$

15. Find the global parametrization of the plane curve,oriented by $\frac{\nabla f}{\|\nabla f\|}$ where f is the function defined by the left side of the equation $x_1^2 - x_2^2 = 1, x_1 > 0$.

16. Let η be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by $\eta = -\frac{x_2}{x_1^2+x_2^2}dx_1 + \frac{x_1}{x_1^2+x_2^2}dx_2$. Let C denote the ellipse $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$ oriented by its inward normal, evaluate $\int_C \eta$.

17. *Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ let $p \in S$ and let $\{k_1(p), k_2(p), \ldots, k_n(p)\}$ be the principal curvatures of S at p with corresponding orthogonal principal curvature directions $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Prove that the normal curvature $k(\mathbf{v})$ in the direction $\mathbf{v} \in S_p$ is given by*
$$k(\mathbf{v}) = \sum_{i=1}^{n} k_i(p)(\mathbf{v} \cdot \mathbf{v}_i)^2 = \sum_{i=1}^{n} k_i(p)cos^2\theta_i \quad \textit{where } \theta_i = cos^{-1}(\mathbf{v} \cdot \mathbf{v}_i) \textit{ is the angle between } \mathbf{v} \textit{ and } \mathbf{v}_i.$$

18. a) Define coordinate vector fields along a smooth map $\varphi : U \to \mathbb{R}^{n+k}$, where U open in $\mathbb{R}^n$.
    b) Find the coordinate vector fields along the parametrized torus φ in $\mathbb{R}^3$ given by
    $\varphi(\theta, \phi) = ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi)$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. a) Is there any relation between level sets and graph of a given function defined on $U \subseteq \mathbb{R}^{n+1}$. Explain.
    b) Show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$.
    c) Obtain level sets and graph of the function $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \ldots + x_{n+1}^2$ for $n = 0, 1$

20. Given S is a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0, \forall p \in S$. Is the Gauss map from S to the unit sphere S^n onto ? Explain.

21. For the Weingarten map L_p, prove that $L_p(\mathbf{v}).\mathbf{w} = \mathbf{v}.L_p(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in S_p$.

22. *Let S be an oriented n-surface in $\mathbb{R}^{n+1}$ and let $\mathbf{v}$ be a unit vector in S_p, $\mathbf{p} \in S$. Prove that there exists an open set $V \subset \mathbb{R}^{n+1}$ containing $\mathbf{p}$ such that $S \cap \mathcal{N}(\mathbf{v}) \cap V$ is a plane curve. Moreover, the curvature at $\mathbf{p}$ of this curve (suitably oriented) is equal to the normal curvature $k(\mathbf{v})$.*

(2×5=10 weightage)