QP CODE: 25024352

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Reg No : ....

Name : ....

M.Sc DEGREE (CSS) EXAMINATION, APRIL 2025

Fourth Semester

M Sc MATHEMATICS

CORE - ME010402 - ANALYTIC NUMBER THEORY

2019 ADMISSION ONWARDS

C7449C4F

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. Define the identity function $I(n)$ and prove that $I * f = f * I = f$, where f is any arithmetical function.
2. Prove that if f and g are multiplicative then so is their Dirichlet product.
3. Define the divisor function $\sigma_\alpha(n)$. Show that it is multiplicative.
4. Prove that the following relations are logically equivalent.
   (a) $\lim_{x\to\infty} \frac{\pi(x) log x}{x} = 1.$
   (b) $\lim_{x\to\infty} \frac{\pi(x) log \pi(x)}{x} = 1.$
5. Prove that $\forall\, x \geq 1, \sum_{n\leq x} \frac{\Lambda(n)}{n} = log\, x + O(1).$
6. State and prove cancellation law.
7. Give an example of a linear congruence having no solution.
8. Solve the congruence $5x \equiv 3 (mod 24)$.
9. Define quadratic residues. Find the quadratic residues for p=11.
10. (a) Define the exponent of a modulo m .
    (b) Let $m \geq 1$ and $(a, m) = 1$. Then prove that $a^k \equiv 1 (mod m)$ if and only if $k \equiv 0 (mod\, m)$, where $f = exp_m(a)$.

(8×1=8 weightage)

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then prove that $\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t - [t])f'(t)dt + f(x)([x] - x) - f(y)([y] - y)$.

12. If $h = f * g, H(x) = \sum_{n\leq x} h(n), F(x) = \sum_{n\leq x} f(n), G(x) = \sum_{n\leq x} g(n)$, prove that $H(x) = \sum_{n\leq x} f(n)G(\frac{x}{n}) = \sum_{n\leq x} g(n)F(\frac{x}{n})$.

13. Prove that $\lim_{x\to\infty} (\frac{\psi(x)}{x} - \frac{\vartheta(x)}{x}) = 0$.

14. If $n \geq 1$,show that $\frac{1}{6}n\,log\,n < P_n < 12(n\,log\,n + n\,log\frac{12}{e})$, where P_n denotes the n^{th}prime .

15. For any prime $p \geq 5$, prove that $\sum_{k=1}^{p-1} \frac{(p-1)!}{k} \equiv 0\ (mod\,p^2)$.

16. Let f be a polynomial with integer coefficents, let $m_1, m_2, ..., m_r$ be positive integer relatively prime in pairs, and let $m = m_1 m_2 \ldots m_r$ Then prove that the congruence $f(x) \equiv 0(mod\ m)$ has a solution if and only each of the congruence $f(x) \equiv 0(modm_i)$, i=1,,2,...,r , has a solution.

17. Determine those odd primes p for which 3 is a quadratic nonresidue.

18. Let $(a, m) = 1$ and $f = exp_m(a)$. Prove that $exp_m(a^k) = exp_m(a)$ if and only if $(k, f) = 1$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. Prove that (a)a lattice point (a, b) is visible from the origin if and only if a and b are relatively prime.
(b)If the two integers a and $b$are chosen at random, then the probability that they are relatively prime is $\frac{6}{\pi^2}$.

20. State and prove Shapiro's Taubarian Theorem.

21. State and prove the Chineese remainder theorem. Hence prove that the linear system of congruences $a_1 x \equiv b_1(mod\,m_1), \ldots, a_r x \equiv b_r(mod\,m_r)$ has exactly one solution modulo $m_1, \ldots, m_r$. Asume that $m_1, \ldots, m_r$ are positive integers, relatively prime in pairs, $b_1, \ldots, b_r$ be arbitrary integers and $a_1, \ldots, a_r$ satisfy $(a_k, m_k) = 1$ for$k = 1, \ldots, r$.

22. If p and q are distinct odd primes prove that $(p|q) = \begin{cases} -(q|p) & \text{if } p \equiv q \equiv 3\ mod\ 4 \\ (q|p) & \text{if otherwise} \end{cases}$

(2×5=10 weightage)